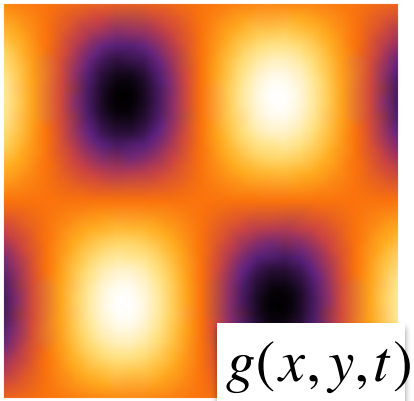
Image Data Compression

Introduction to Coding

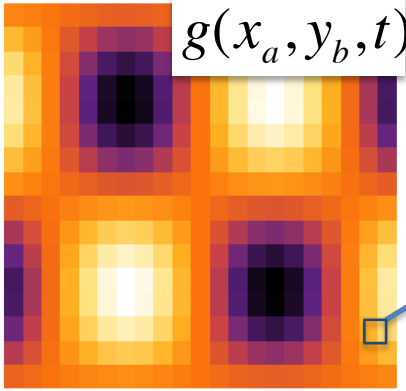
Review: data reduction steps (discretization / digitization)

Continuous 2D signal
(light intensity on sensor)



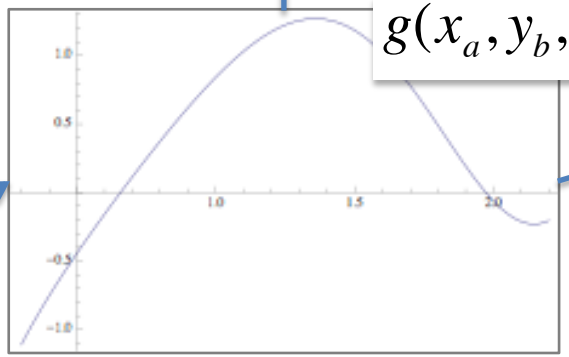
$g(x, y, t)$

Spatial discretization



$g(x_a, y_b, t)$

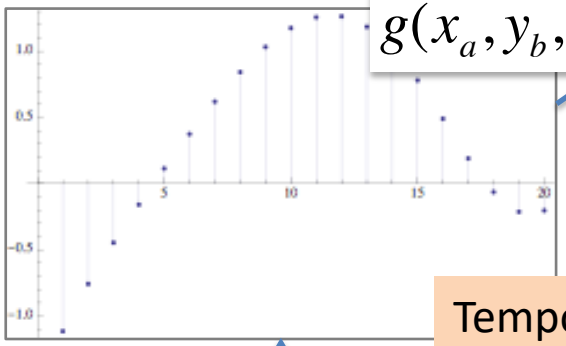
Spatially discrete signal
(pixel-averaged intensity)



$g(x_a, y_b, t)$

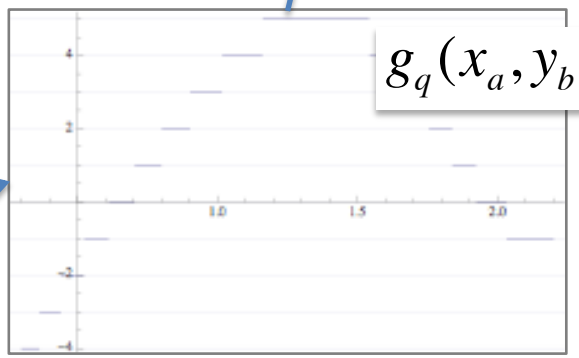
Analog signal
(light intensity at a pixel)

Discrete time signal
(pixel voltage readings)



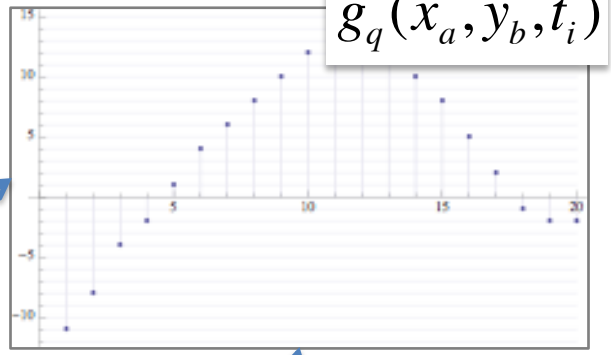
$g(x_a, y_b, t_i)$

Temporal discretization and digitization



$g_q(x_a, y_b, t)$

Discrete value signal
(e.g., # of electrons at each pixel of the CCD matrix)



$g_q(x_a, y_b, t_i)$

Fully digital signal

Review: data reduction steps (discretization / digitization)

Discretization of 1D continuous-time signals (sampling)

- Important signal transformations: up- and down-sampling
- Information-preserving down-sampling: rate determined based on signal bandwidth
- Fourier space allows simple interpretation of the effects due to decimation and interpolation (techniques of up-/down-sampling)

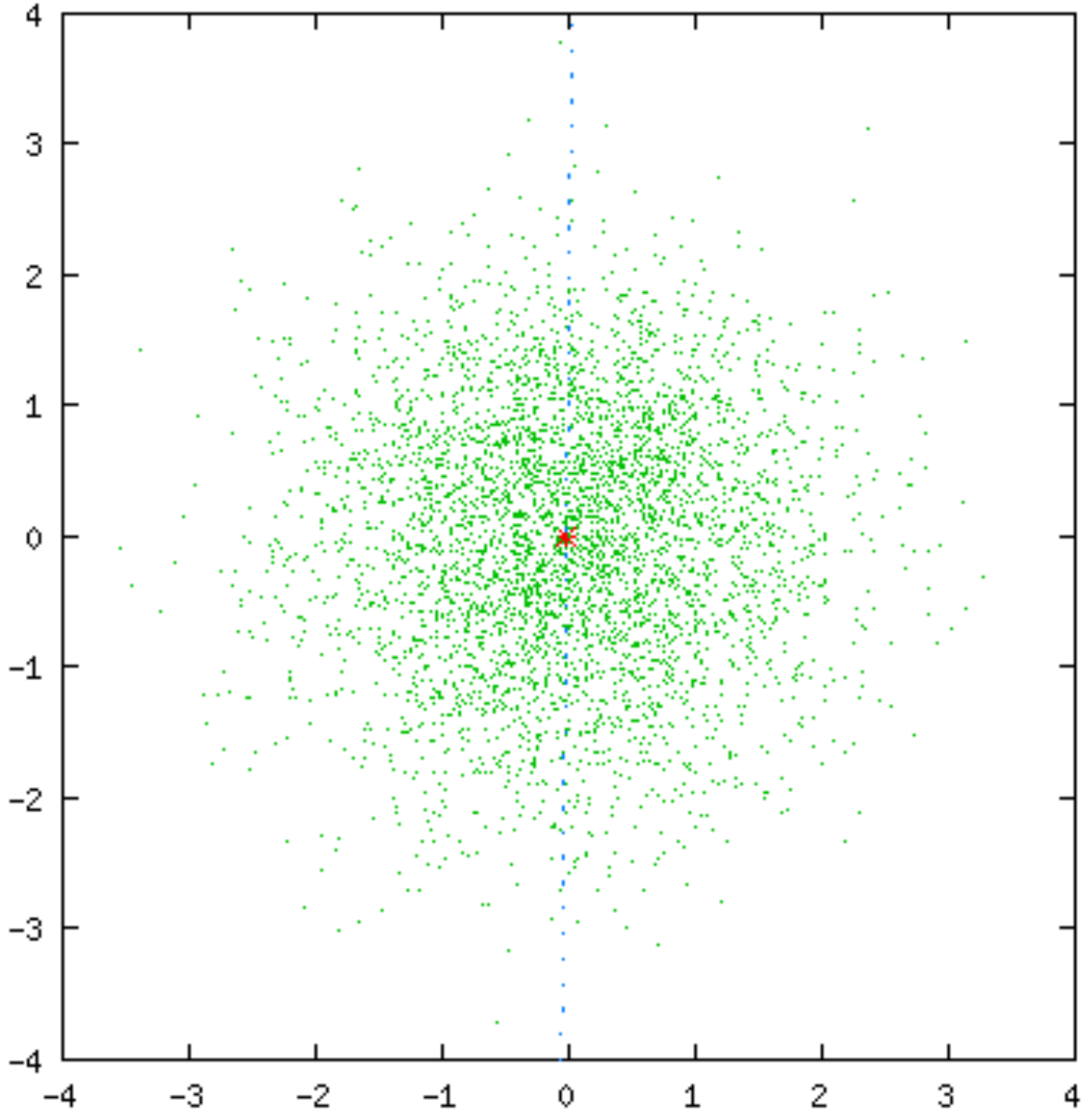
Scalar (one-dimensional) signal quantization of continuous-value signals

- Quantizer types: uniform, simple non-uniform (with a dead zone, with a limited amplitude)
- Advanced quantizers: PDF-optimized (Max-Lloyd algorithm), perception-optimized, SNR-optimized
- Implementation: pre-processing with a compander function + simple quantization

Vector (multi-dimensional) signal quantization

- Terminology: quantization, reconstruction, codebook, distance metric, Voronoi regions, space partitioning
- Relation to the general classification problem (from Machine Learning)
- Linde-Buzo-Gray algorithm of constructing (sub-optimal) codebooks (aka k-means)

LGB vector quantization – 2D example



[Linde, Buzo, Gray '80]:

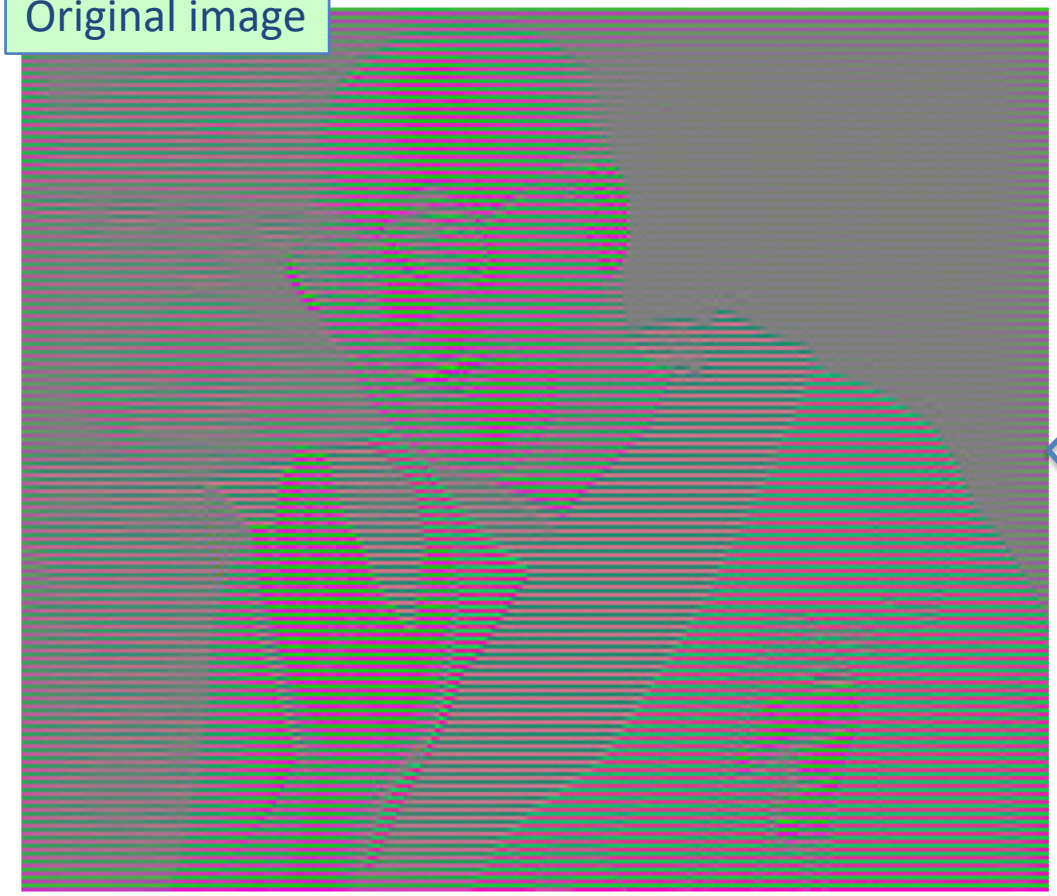
1. Choose codebook size N
2. Select N random codewords as the initial codebook Y
3. Classify training vectors (i.e. find V_i for each x_k)
4. Update codewords: in each cluster,

$$\vec{y}_i \leftarrow \frac{1}{N_i} \sum_{\vec{x}_k \in V_i} \vec{x}_k$$

5. Repeat steps 2 and 3 until convergence (i.e. until changes in codewords become small)

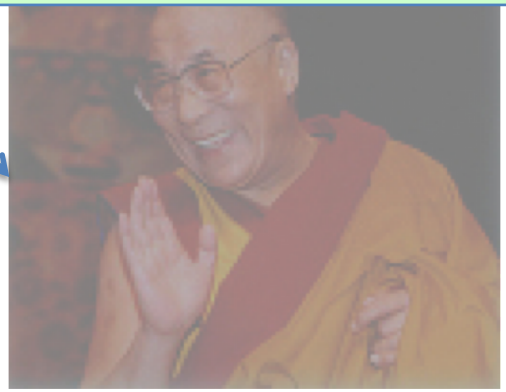
Example: gamma-error in picture scaling [Brasseur 2007]

Original image



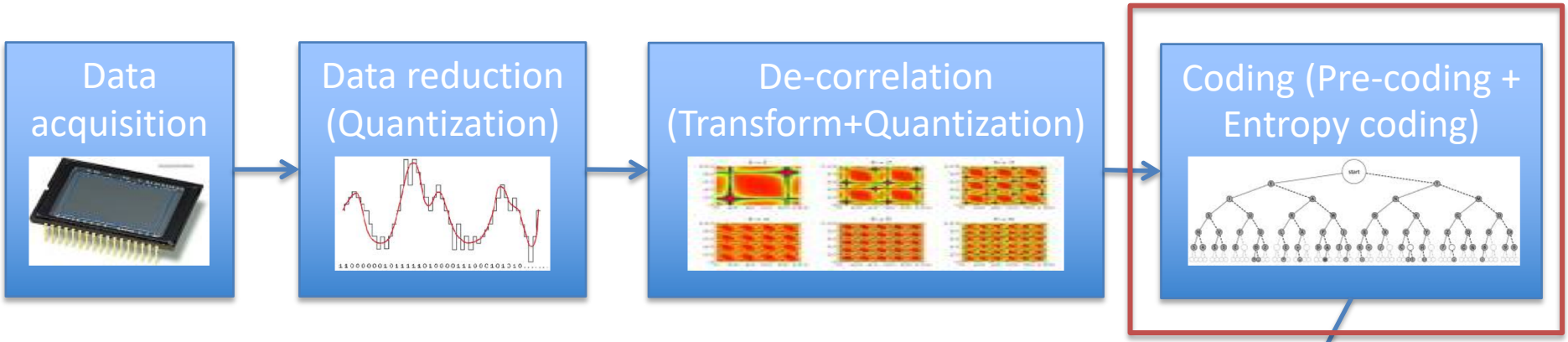
Resized image (averaging of indices)

Resized image (averaging of values)



Gamma-correction:

Image signal path: overview



Output data:
Continuous multi-dimensional analog signal

Compression:
Physical sensor limitations, DAQ settings

Output data:
Correlated fully digital multi-dimensional symbol stream

Compression:
Spatio-temporal quantization, value quantization

Output data:
Linear stream of de-correlated symbols

Compression:
Truncation of irrelevant signal parts, value quantization

Output data:
Linear stream of bits

Compression:
Reversible suppression of the redundant information

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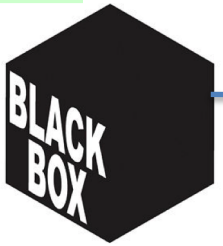
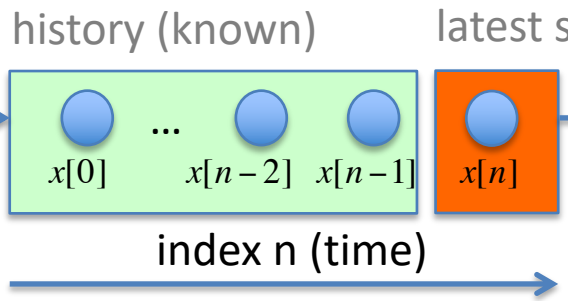
(Reversibly) coding finite strings: basic definitions

Source: produces a sequence of symbols



E.g., image data in some representation



Coder

(ideally, would like to code only the new information!)

Intuitively: how much new information does $x[n]$ bring?
It depends on how well we can predict $x[n]$, given the **a priori knowledge** and the available **history** ($x[0]$, …, $x[n-1]$).

Common notation

- Input **symbols** (characters, events): s_i
- Input **alphabet**: set of all possible distinct input symbols: $Z = \{s_i\}, \quad i = 1,\ldots,K$
- **Signal**: time-discrete finite sequence of symbols: $x[n] \in Z$, n is the time moment
  (Slight abuse of notation: $x[n]$ may denote the n-th value as well as the entire finite sequence $x[0]$ … $x[n]$)

- **Model**: probability that symbol s_i appears at the time moment n, given the entire signal history:

$$p_i^{(n)} \equiv p(x[n] = s_i \mid x[n-1],\ldots,x[0])$$

Coding finite strings: basic definitions (following [Shannon 1948])

Alphabet: $Z = \{s_i\},\ i = 1,\ldots,K$ **Signal:** $x[n] \in Z$ **Model:** $p_i^{(n)} \equiv p(x[n] = s_i \mid x[n-1],\ldots,x[0])$

[Ergodic] stochastic process, modeled as a Markov chain (a special type of Bayesian Networks):

0-th order model

$$p_i^n = 1/K$$

(no a priori knowledge)

Example: English, 27 letters: "xfoml rxkhrjffjuj zlpwcfwkcyj ffjeyvkcqsghyd qpaamkbzaacibzlhjqd"

1-st order model

$$p_i^{(n)} = p(x[n] = s_i) = p_i$$

(symbols are statistically independent, we account only for the frequency of individual symbols)

"ocroh hli rgwr nmielwis eu ll nbnesebya th eei alhenhttpa oobttva nah brl"

2-nd order model

$$p_i^{(n)} = p(x[n] = s_i \mid x[n-1])$$

(pairwise dependencies are important, need individual symbol frequencies and frequencies of digrams)

"on ie antsoutinys are t inctore st be s deamy achin d ilonasive tucoowe at teasonare fuso tizin andy tobe seace ctisbe"

3-rd order model

$$p_i^{(n)} = p(x[n] = s_i \mid x[n-1], x[n-2])$$

(dependence of each symbol on two previous symbols, i.e. add trigram statistics)

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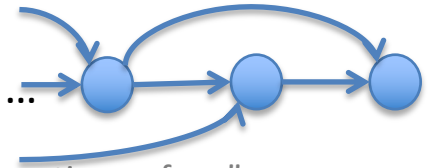

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Fully general model: infinite order (dependence of each symbol on all previous symbols)

Codeword-based coding of symbol strings: basic definitions

Alphabet: $Z = \{s_i\},\ i = 1,\ldots,K$ **Signal:** $x[n] \in Z$ **Model:** $p_i^{(n)} \equiv p(x[n] = s_i \mid x[n-1],\ldots,x[0])$

- Given a model, the **information content** of the actual symbol s_i is :
  ("the more surprising a symbol, the more information is brings")

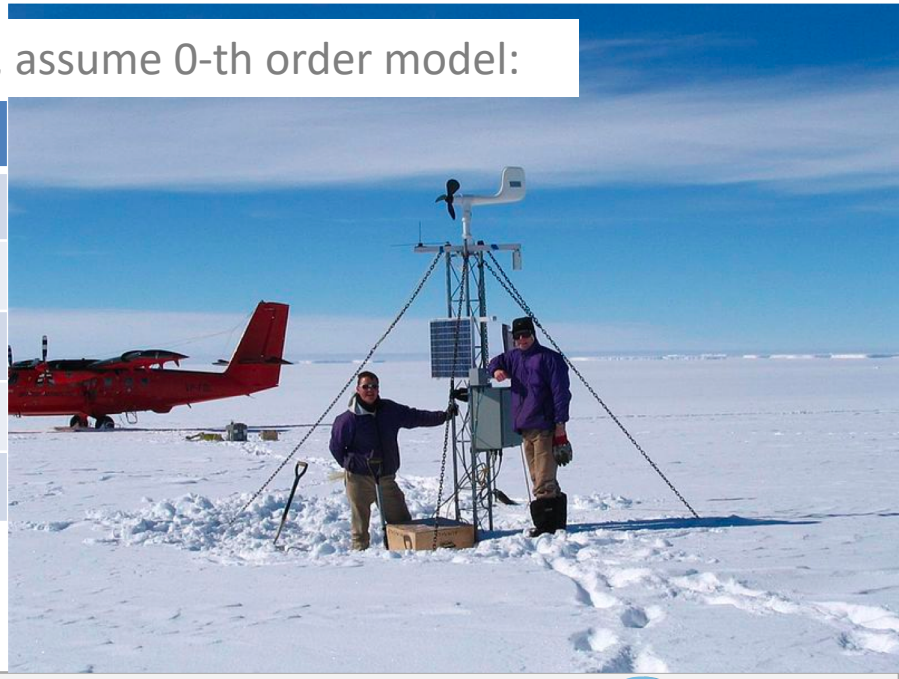
$$I(s_i) = -\log_2(p_i^{(n)}) \cdot [bit]$$

- **Codeword-based coding**: substitute each symbol with a **codeword**.
- Example: symbol **"A"** maps to the binary codeword **"01101"**, codeword length: 5 bits.
  (here we only discuss binary codes; relation to other bases can be easily established)

- **Code**: set of all codewords for an alphabet

Example: weather observations at a remote station, assume 0-th order model:

Weather	P = 1/K	I[bit]	Codeword
Sun	0.25	2	00
Clouds	0.25	2	01
Rain	0.25	2	10
Snow	0.25	2	11
	$\Sigma\ p_i = 1.0$		



Information content of each symbol matches its code length, no shorter representation

1-st order process model (entropy coding), Shannon bound

Alphabet: $Z = \{s_i\},\ i = 1,\ldots,K$ **Signal:** $x[n] \in Z$ **1-st order model:** $p_i^{(n)} = p_i$

- We always consider a "long enough" series of observations (n -> ∞)
- Assume an **ergodic source**: symbol statistics in a string approach those over all strings
- **Average codeword length**:

$$S_{src} = \langle l_i \rangle = \sum_{i=1}^{K} p_i \cdot l(s_i) \cdot [bit / symbol]$$

- **Source entropy**: average information content of a symbol (e.g. over any infinite string):

$$H_{src} = \langle I(s_i) \rangle_Z = \sum_{i=1}^{K} p_i \cdot I(s_i) = -\sum_{i=1}^{K} p_i \log_2(p_i) \cdot [bit / symbol]$$

- **[Shannon '48], [Shannon, Weaver '49]: average length of any entropy code is limited by H_{src}. No shorter code can be decoded unambiguously.** (We will not prove this theorem here.)

$$H_{src} \le S_{src}$$

- **Code redundancy**:

$$\Delta R = S_{src} - H_{src} \ge 0$$

Example: unary coding (assume a 0-th order model)

Alphabet: all decimal digits. A digit q is coded as “1” repeated q times and a terminal “0”:

Symbol	Code
0	0
1	10
2	110
3	1110
4	11110
5	111110
6	1111110
7	11111110
8	111111110
9	1111111110

Assume the equal probability of all ten digits:

$$p_i^{(n)} = p_i = \frac{1}{10}, i = 1,\ldots,10$$

Average codeword length for unary coding:

$$S_{src} = \langle l_i \rangle = \sum_{i=1}^{10} \frac{1}{10} \cdot (i+1) = 5.5$$

Source entropy:

$$H_{src} = \langle I(s_i) \rangle_Z = -\sum_{i=1}^{10} \frac{1}{10} \cdot \log_2\left(\frac{1}{10}\right) = 3.32$$

Code redundancy:

$$\Delta R = S_{src} - H_{src} = 2.17[bits/digit]$$

Example: binary coding (assume a 0-th order model)

Regular binary coding:

Symbol	Code
1	000
2	001
3	010
4	011
5	100
6	101
7	110

- Optimal if the alphabet size is a power of 2
- Wasted codeword length otherwise!
- (here: codeword “111” is not used)

Slightly more efficient: truncated binary coding

Given symbol d (d = 0, …, m - 1), compute

$$b = \lceil \log_2 m \rceil, \quad t = 2^b - m .$$

Then: if d < t, output (b - 1)-bit binary representation of d

Otherwise: output b-bit binary representation of (t + d)

- Can be decoded (longer codewords start with unused binary sequences)
- Optimal for the uniform probability distribution

$$H_{src} = 3.32[\text{bits/digit}]$$

$$S_{src} = 3.4[\text{bits/digit}]$$

b = 4

t = 6

m = 10

Symbol	Code
0	0 0 0
1	0 0 1
2	0 1 0
3	0 1 1
4	1 0 0
5	1 0 1
6	1 1 0 0
7	1 1 0 1
8	1 1 1 0
9	1 1 1 1

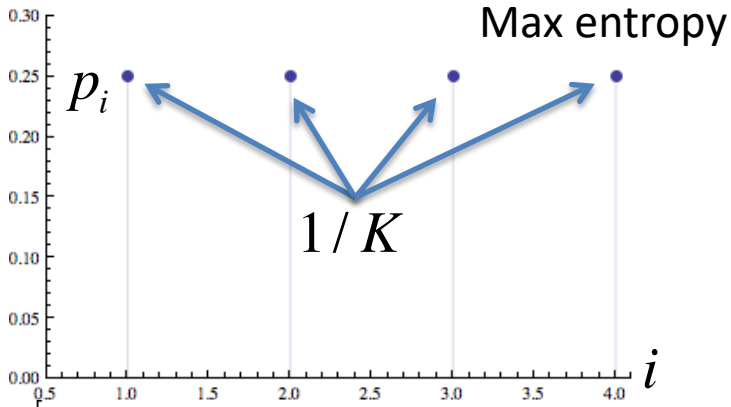
(b - 1) bits

b bits

Properties of entropy (discrete probabilities)

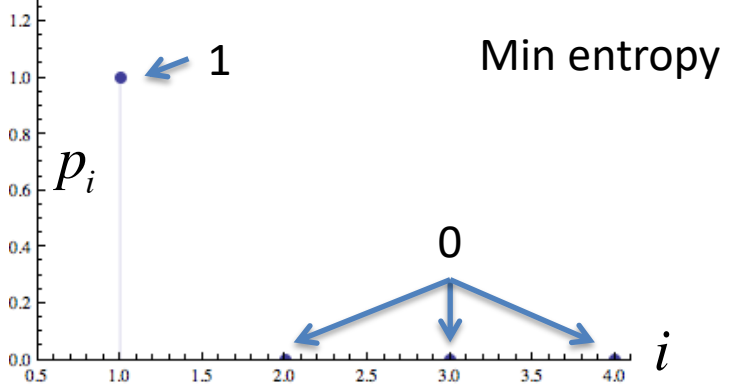
- Uniform distribution has the maximal entropy:

$$0 \le H \le H_{\max} = \sum_{i=1}^{K} \frac{1}{K} \log_2 K = \log_2 K$$



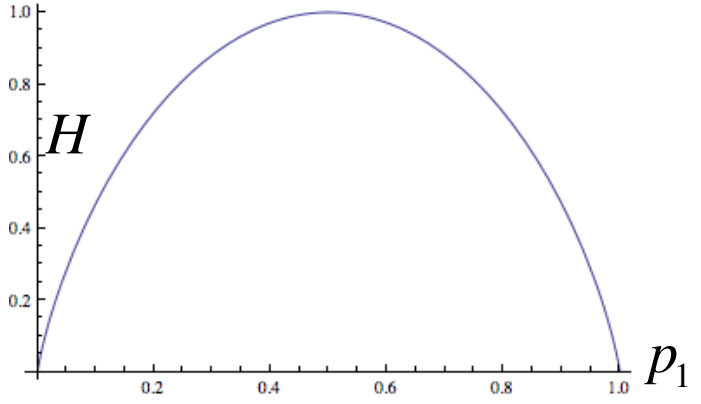
- If the next incoming symbol can be predicted (complete deterministic knowledge), one symbol has 100% probability, and the source entropy is zero:

$$H = 0 \Leftrightarrow p_i = \delta_i^j$$



- Special case: binary alphabet (two symbols):

$$H = -p_1 \log_2 p_1 - (1 - p_1) \log_2 (1 - p_1)$$



In other words, entropy is a measure of uncertainty

Entropy coding: process models of the 1-st order

Example: a 1-st order model, weather observation data:

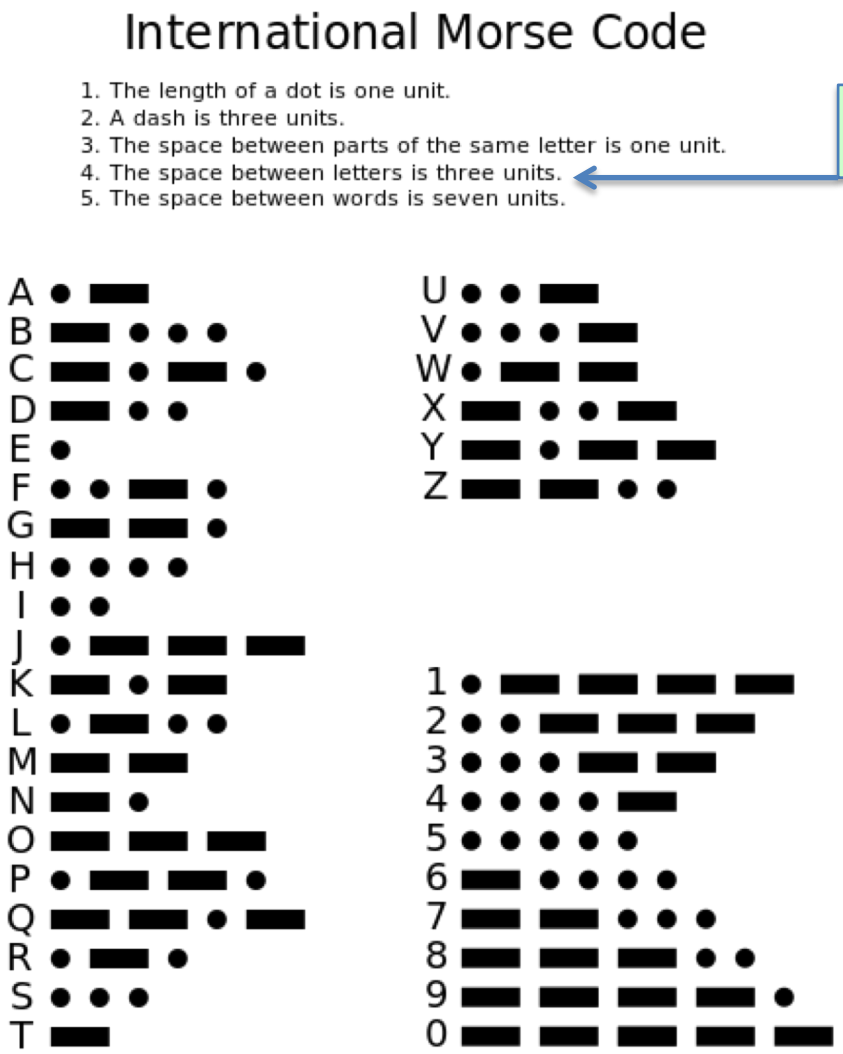
Weather	P	I[bit]	Codeword
Sun	0.5	1	00
Clouds	0.25	2	01
Rain	0.125	3	10
Snow	0.125	3	11
	$\Sigma\ p_i = 1.0$	$H_{src} = 1.75$	$S_{src} = 2$

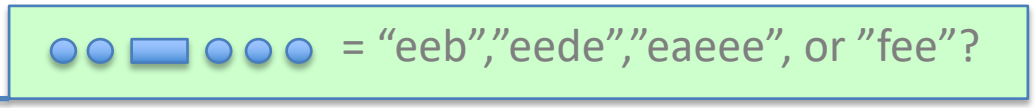
A fixed length code, always uses 2 bits per symbol

Entropy H_{src} = 1.75 bits/symbol, average codeword length S_{src} = 2 bits/symbol
Redundancy: ΔR = 0.25 bits/symbol

Obvious questions:
1. Can we do better?
2. If yes, then by how much exactly better?

Variable-length prefix codes: Morse code [1843]





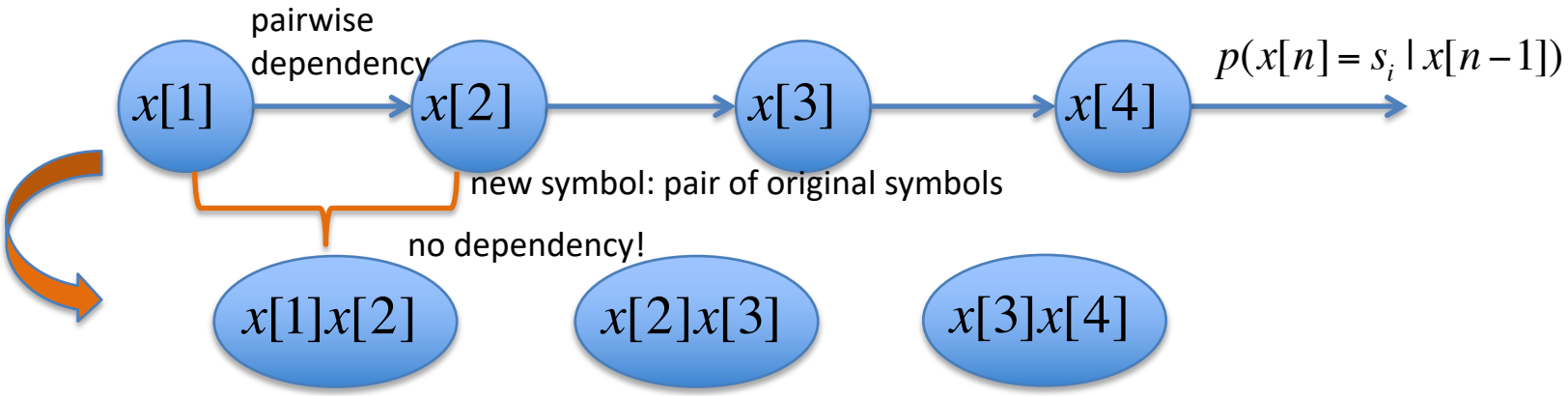
- Must necessarily be uniquely decodable!
- Ternary code: Z = {dot, dash, “3-unit pause”}
- Binary code: Z = {“low”, “high”}, and “3-unit pause” as obligatory suffix to each codeword
- Letter probabilities are different, so are the codeword lengths: spares time!
- First-order process model: most common letters (E, T) receive shorter codes, least common (J, Q, X, Z) – longer codes.
- Rudimentary second-order elements: e.g. a “space” cannot follow another “space”

Prefix-free (=prefix, =instantaneous) codes: no codeword is a prefix for another codeword

Later we will study codes which very efficiently solve the entropy coding problem!

Reducing an M-th order problem to a 1-st order problem

Example: 2-nd order Markov chain reduced to a 1-st order problem:



$$p(x[n]x[n-1] = s_i s_j) = p(x[n] = s_i \mid x[n-1] = s_j)\, p(x[n-1] = s_j),$$

having received x[n-1]: $p(x[n-1] = s_j) = 1$

However, this is not always practical:

- Alphabet size grows exponentially with *n*-gram length (i.e. as K^M)
- Higher order conditional probabilities become less accurate
  (need more training data to determine the probabilities of all new symbols)
- Not all data types are well-represented by finite (or lower-) order Markov processes:

> "three Swedish witches watch three Swiss Swatch watch switches.
> Which Swedish witch watches which Swiss Swatch watch switch?"

- very repetitive (i.e. compressible), but typical inter-letter correlation length is larger than e.g. 2

- **Instead:** perform data-matched de-correlation (transform), then apply entropy coding

General models: guessing inter-symbol dependencies

- Consider general (∞-order) model with complete dependencies on previous values:

$$p_i^{(n)} = p(x[n] = s_i \mid x[n-1], \ldots, x[0])$$

- Need to consider probabilities of all finite (sub)strings: $p(x[n], x[n-1], \ldots, x[0])$
- If we allow dependences between symbols, we might define "conditional entropy":
  (precise definition will be given later) $H_{cond} \le H_{src}$
- Define inter-symbol redundancy:
  (how much do we win by considering inter-symbol correlations?)

$$\Delta R_{cond} = H_{src} - H_{cond}$$

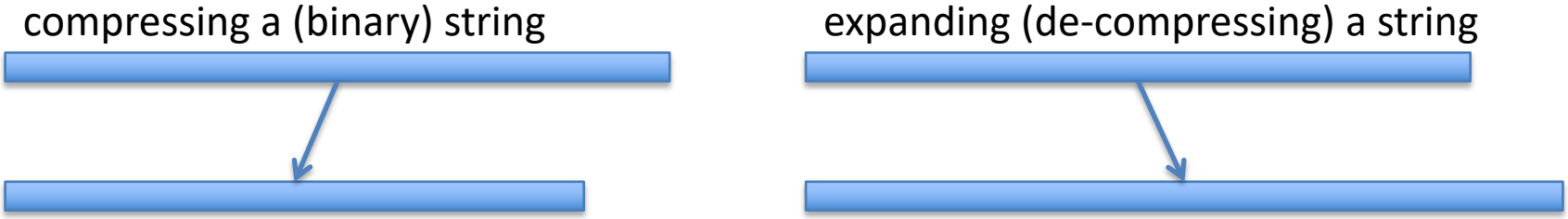
- **Modeling**: reduce (original) inter-symbol redundancy (after coding), optimally to zero
  - We cannot find the optimal solution (i.e. there exists no universal compression)
  - In general, an AI problem (and domain-dependent art)
  - There exists universal but uncomputable model (probability distribution) of data.

Example: let us code the digits of π = "3.14159265358979323846264338327950288419716…"

- As 0-th order model: 10 independent digits, $p_0 = p_1 = \ldots = p_9 = 0.1,$
  Shannon bound: $H_{src} \approx 3.3219$ [bits/digit], need > 415241 bytes for first million digits
- As a string: *"the first million digits of pi"* = 30 bytes, 0.00024 [bits/digit]!

Alternatively: write a program to compute π to any arbitrary length: up to 52 bytes in size!

General models: universal compression does not exist



- **Counting argument**: there are 2^n strings of length n, but only $\sum_{i=0}^{n-1} 2^i = 2^n - 1$ strings shorter than n; ➔ no bijective mapping that reduces all strings.
- Fraction of strings of length n that can be compressed to m bits: at most $2^{(n-m)}$
- **Example**: fraction of strings (of any length) compressible by 1 byte is $\mathbf{2^{-8} = 0.4\%}$
- Every compressor *must* expand some inputs. However, expansion never needs to add more than one bit (e.g. return the unchanged input string + 1-bit indicator flag).
- **Recursive compression is prohibited**: optimally compressed data appears random to any algorithm, cannot be compressed further (but sub-optimally compressed data may be).

A Very Remarkable and Non-trivial Experimental Fact of Nature:
- most strings we care about (text, images, movies, software, DNA, …) are compressible!

Optimal modeling is not computable

[Solomonoff 1960, 1964], [Kolmogorov 1965], [Chaitin 1966]:

Algorithmic Probability: a universal (!) a priori (!) probability distribution over any strings

- Given a string x, compute following the sum over all programs M in language L that output x, with summand being $2^{-|M|}$, where $|M|$ is length of program M:

$$m(x) := \sum_{M:U(M)=x} 2^{-|M|}$$

Kolmogorov complexity of string x: $K_L(x) = |M|_{min}$; $m(x)$ above is dominated by $2^{-K(x)}$

- Invariance theorem: $K_L(x)$ in languages L_1 and L_2 differ only by an x-independent constant
  (Proof: given description M_1 in language L_1, output string x and append compiler from L_1 to L_2)

Best compression we can achieve for any string *x* is encoding it as the shortest program *M*

- Incomputability of $|M|_{min}$: assume an algorithm to find $|M|_{min}$ exists, enumerate its outputs; then *"the first string that cannot be described in million bits"* leads to a paradox.

Some minor remarks:

- It is not yet proven that the algorithmic probability is the true universal prior probability, but it is a widely accepted statement on empirical grounds
- **Minimal Description Length (MDL)** principle in Machine Learning: choose the simplest hypothesis that is consistent with the data
- **Occam's Razor** (14th century), basis of science: "choose the simplest answer that explains all facts"

Universal randomness test does not exist (again, no proofs)

[Gödel 1931]: first incompleteness theorem:
In any consistent formal system (set of axioms and rules of inference) powerful enough to describe rules of arithmetic, there is at least one true statement that cannot be proven.

[Li, Vitanyi 2007]: stronger variation of the Gödel's theorem
If the system is sound (cannot prove false statements), then there is an infinite number of true but un-provable statements. In particular, there is only a finite number of provable statements of type "x is random" ($K_L(x) \geq |x|$) and an infinite number of finite strings x.

Proof: suppose otherwise. Then one can enumerate all proofs (lists of axioms and applications of inference rules) and describe the paradoxical string

S = *"string x such that it is the first string to be proven to be a million bits long and random"*.

Similarly, there is no test to determine whether a compressed string can be compressed further (since this is equivalent to testing the compressed string for randomness).

Moral to take home: optimal compression is uncomputable.

A relatively “hard” example of a non-random string

Recipe to cook a non-random string:

- Take a string of million zero bytes (very non-random!)
- Encrypt it with the AES cryptographic algorithm in CBC mode with the key “foo”
- The actual result (a few first bytes, 1000032 bytes in total):

“

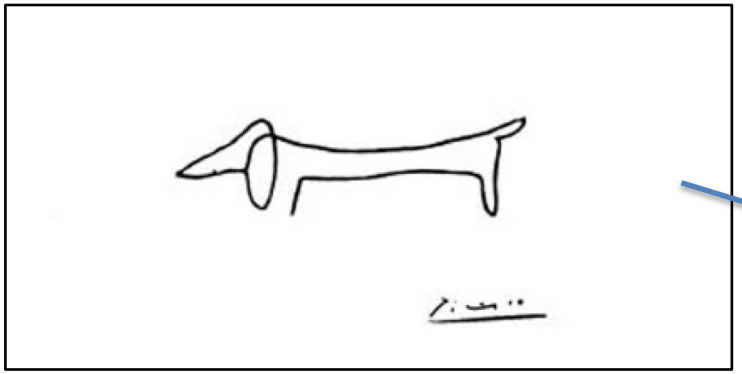
53 61 6C 74 65 64 5F 5F F0 A4 5F 70 56 1F 61 9B
9B 64 53 37 EF C6 22 D3 73 75 9E 46 F9 10 74 A6
9C 9A D5 08 F7 D8 33 A3 B7 91 9C D0 C3 CE D8 ED
39 F8 1A 63 F3 23 CB CF 5D 8C CC 14 8A 32 0C 58
5B 53 55 0A 6A 15 AC 84 8E EE D0 71 9D 9C A7 DD
...

“

- Looks (at least to me) pretty random (due to the design of the cryptographic algorithm), and incompressible almost to any algorithm [unless you know the key]!

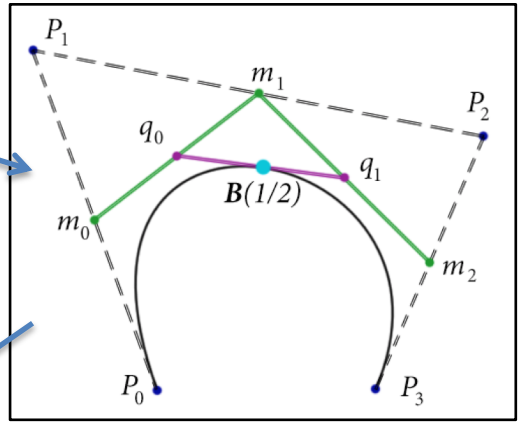
Example from image compression [Kun 2013]

"Dog" sketch by Picasso (costs many $M):

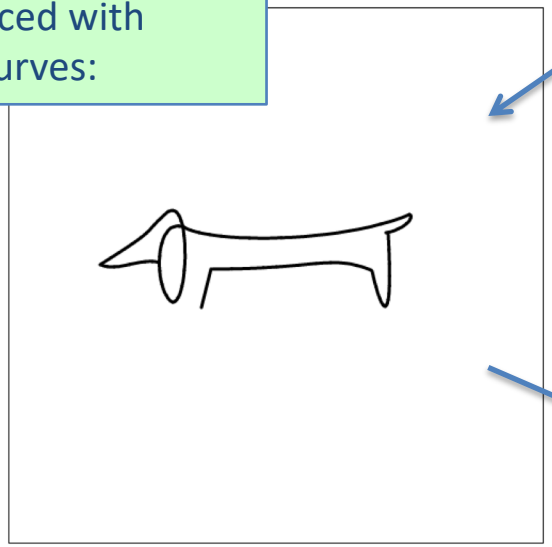


File in some graphics format, a few MB

Bezier polynomial representation of smooth curves (using a few key points and parameters):



The same drawing reproduced with Bezier curves:



Parameters of the resulting Bezier polynomials:

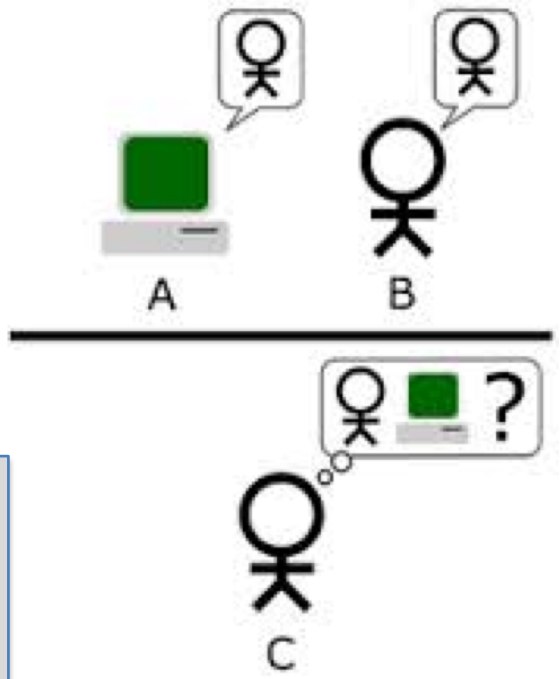
```
500 500
[180,280] [183,268] [186,256] [189,244] // front leg
[191,244] [290,244] [300,230] [339,245] // tummy
[340,246] [350,290] [360,300] [355,210] // back leg
[353,210] [370,207] [380,196] [375,193] // tail
[375,193] [310,220] [190,220] [164,205] // back
[164,205] [135,194] [135,265] [153,275] // ear start
[153,275] [168,275] [170,180] [150,190] // ear end + head
[149,190] [122,214] [142,204] [85,240]  // nose bridge
[86,240] [100,247] [125,233] [140,238]  // mouth
```

(Almost) equivalent representation, a few bytes!

Modeling of data as an Artificial Intelligence problem

Turing test from 50's on identifying the human intelligence:

- A human judge, human confederate, machine, and terminal.
- The confederate and the machine try to convince the judge that each of them is a human.
- If the judge cannot guess who is a machine correctly in 70% of time after 10 minutes of interaction, then the machine is declared to possess the Artificial Human Intelligence.



A possible transcript of an interaction session:

Q: Please write me a sonnet on the subject of the Forth Bridge.
A: Count me out on this one. I never could write poetry.
Q: Add 34957 to 70764.
A: (Pause about 30 seconds and then give as answer) 105621.
Q: Do you play chess?
A: Yes.
Q: I have K at my K1, and no other pieces. You have only K at K6 and R at R1. It is your move. What do you play?
A: (After a pause of 15 seconds) R-R8 mate.

To fool the judge, the machine needs to predict the human's answer to any given question.

Modeling of data as an Artificial Intelligence problem

- Prediction is intuitively related to understanding:
- *"Sic transit _____ mundi"* – fill in the blanks (if you can)

- [Legg, Hutter 2006]: optimal compression, if it were computable, would optimally solve AI problem under Turing test (1950) and the (more general) universal intelligence test.

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To compress the transcript above, we need a model:

response

context up to the current question

$$p(A \mid Q) = \frac{p(QA)}{p(Q)}$$

Can you guess, what could possibly be hidden behind this bar?

But: exactly the same model is needed to generate responses indistinguishable from those of a human, i.e. pass the Turing test!

Coding problem – summary (I)

- **Coding problem – general framework**
  - We consider a source that emits symbols $x[n] = s_i, S = \{s_0, \dots, s_{K-1}\}$
  - Coding: translate source symbols to a stream of bits (binary output alphabet)
  - Goal: reduce the redundancy (i.e. translation must be reversible and compact)
  - Stochastic approach: source is fully characterized by a stationary probabilistic model

$$p_i^{(n)} \equiv p(x[n] = s_i | x[n-1], x[n-2], \dots, x[0]).$$

- **Shannon [1948]: classification of models by the Markov process order**
- **1-st order processes (the most practically important case): $p_i^{(n)} = p_i$**
  - A very strong requirement on source statistics: no inter-symbol dependencies
  - In other words: $x[n]$ is independent of $x[m]$ for all $n \neq m$.
  - For a 1-st order model, we may define source entropy: $H_{src} = -\Sigma\, p_i \log_2(p_i)$
  - Codeword-based coding: given s_i, output some fixed binary string (a codeword)
  - For any coding scheme, average codeword length is $S_{src} = \Sigma\, p_i\, l_i$, where l_i is the length of a codeword (in bits) corresponding to the input symbol s_i.
  - Shannon boundary: $S_{src} \geq H_{src}$ for any reversible coding scheme.
  - Entropy coding schemes: coding under the 1-st order process assumption
  - There exist very efficient and universal algorithms for entropy coding

Coding problem – summary (II)

- **2-nd order processes:** $\boldsymbol{p_i^{(n)} = p(x[n] = s_i | x[n-1]) = \frac{p(x[n]=s_i, x[n-1]=s_j)}{p(x[n-1]=s_j)} = \frac{p_{ij}}{p_j}}$
  - A slightly more general model, slightly weaker independence assumptions
  - Any 2-nd order process can be represented as a 1-st order process!
  - Cost: growth of the alphabet size (digrams; K^2 symbols instead of K)

- **3-rd order, 4-th order processes, etc.**
  - Can be reduced to a 1-st order model by the same trick
  - Exponential alphabet size growth (i.e. need more data to determine probabilities)
  - Many important data types correspond to models of a very high finite order!
  - In practice, use many ad-hoc (heuristic) algorithms fitted to specific data types
  - "Pre-coding" step (reduces inter-symbol dependencies), then entropy coding

- **General coding problem: an ∞-th order process**
  - For simplicity, consider binary input (i.e. code binary strings -> binary strings)
  - A general model is equivalent to assigning probabilities to all finite binary strings:

$$p_i^{(n)} \equiv p(x[n] = s_i | x[n-1], x[n-2], \dots, x[0]) = \frac{p(x[n], x[n-1], \dots, x[0])}{p(x[n-1], \dots, x[0])}$$

Coding problem – summary (III)

- **General coding problem: ∞-th order process (continued)**
  - Coding: mapping a binary string x to a binary string y. Compression: $|y| < |x|$.
  - Counting argument: universal compression is impossible (some inputs are extended)

- **General coding problem: what is the best compression for a given string x?**
  - Amazingly, there exists a universal a priori PDF $p(x) = p(x[n], \dots, x[0])$
  - Consider all programs M in a language L that output x (i.e. all $M: U(M) = x$)
  - Algorithmic probability of a string x: $p(x) = \sum_{M:U(M)=x} 2^{-|M|}$
  - $p(x)$ is dominated by term 2^{-K_L}, where $K_L(x) = |M|_{min}$ is Kolmogorov complexity (length of the shortest program that outputs x)
  - Shortest program $M: U(M) = x$ is the best compression for string x!
  - Unfortunately, $K_L(x)$ is incomputable (no algorithm to compute $K_L(x)$ may exist)

- **General coding problem: is a given string x compressible?**
  - A string x is random if $K_L(x) \geq |x|$ (i.e. it is the shortest representation of itself)
  - A random string cannot be compressed further (i.e. it is incompressible)
  - Li, Vitanyi, 2007: no algorithm to check if x is random may exist

- **Therefore: optimal compression is impossible!** Only ad hoc data-fitted solutions...
  - Moreover: finding optimal compression = finding universal artificial intelligence